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DYNAMIC PLASTIC RESPONSE OF BEAMS RESTING ON FLUID

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Abstract—This paper considers the two-dimensional problem of a wide rigid ideally-plastic simply-supported beam resting on a semi-infinite pool of potential fluid. It is demonstrated that the problem reduces to determination of the apparent (added, virtual) mass of fluid reflecting inertial characteristics of the fluid resisting deformation. Using the theory of dual integral equations an especially simple closed form solution for the apparent mass is derived enabling solution of this practically significant problem.

NOTATION

$A(a\xi)$	Kernel of integral eqns (23)
a	Half-span of the beam
$F(ar)$	Right-hand side of eqn (23)
$f(x, y)$	Spatial distribution of the velocity potential
$\bar{f}_c(\xi, y)$	Fourier cosine transform of $f(x, y)$
$g(t)$	Auxiliary function (27)
h	Beam height
I	Impulse parameter (49)
J_0	Bessel function of zeroth order
k, k_1	Reduction coefficients (39), (46)
M_f	Angular momentum of fluid back pressure
M_0	Fully-plastic moment in the beam
m_f	Apparent (added, virtual) fluid mass
$m = m_s + m_f$	"Total" mass of the beam-fluid system
m_s	Mass of the beam per unit length
p	Total externally applied load
p_e	External load
p_f	Fluid back pressure
$r = x/a$	Nondimensional coordinate
t	Time variable
t_f	Total deformation time
$u = a\xi$	Nondimensional variable
$W_0(t)$	Displacement of the beam midspan
$w(x, t)$	Beam deformation
x, y	Coordinate frame
α	Parameter (40)
γ_f	Fluid density
γ_s	Density of the beam
μ	Static collapse load parameter (42)
$\theta(t) = W_0(t)/a$	Angle of rotation of the beam about the support
ξ	Transform variable
$\phi(x, y, t)$	Velocity potential
χ, ψ	Auxiliary functions (24)

1. INTRODUCTION

The general problem of dynamic response of rigid ideally-plastic structures has been investigated rather extensively in the past two decades (see, e.g. Ref.[1] and state-of-art reviews[2, 3]). The existing literature offers a variety of exact solutions, approximate methods and bounding techniques applicable to problems involving structures surrounded by vacuum (i.e. low density media).

On the other side the importance of fluid-structure interaction on the system response has long been recognized in aeroelasticity[4], marine engineering[5], and somewhat more recently in reactor engineering[6]. Most, if not all, of the published research centers on perfectly elastic structures. This assumption is reasonably representative of actual material behavior in case of fluid-induced vibrations or stability (flutter) problems. However, stresses in structures submerged into fluid and exposed to transient loads of high intensity and short duration cannot be realistically expected to remain in elastic range. For example, it becomes imperative to estimate residual deformation of a reactor core subassembly duct, associated with a high pressure transient (due to an accidental excursion), in order to assess core alignments critical for subsequent rod insections and withdrawals.

It, therefore, seems worthwhile to study problems of dynamic response of rigid ideally-plastic structures immersed into fluid, subjected to high pressure transients of short duration. The main purpose of this paper is to explore possibilities of solution of such an interaction problem and to offer a qualitative study of a simple problem. It is felt that such a solution will also offer useful guidance in other more complicated problems of practical significance.

Thus, this paper presents a study of an essentially two-dimensional problem of a rigid ideally-plastic simply-supported beam resting on a semi-infinite pool of potential (inviscid and incompressible) fluid.

2. FORMULATION OF THE PROBLEM

Consider a plate of constant thickness and density simply supported at $x = \pm a$ (Fig. 1) and resting on an infinitely extending pool of potential fluid. The plate is subjected to a time-dependent load $p_e(x, t)$ symmetric with respect to x and not depending on coordinate z. Assume the plate, finally, to be infinite in z-direction rendering the problem two-dimensional.

On the assumption of rigid ideally-plastic material behavior, the plate (beam) will not deform until the maximum bending moment at a cross section of the beam equals the fully plastic moment M_0[1, 7].

Assuming triangular deformation mode (Fig. 2) with a plastic hinge at the midspan, the equation of the balance of angular momentum reads

$$-\frac{m_s a^3}{3}\ddot{\theta} = M_0 - \int_0^a (p_e - p_f)(a - x)\,\mathrm{d}x \tag{1}$$

where $\theta(t)$ is the angle of rotation of the rigid beam segment, m_s the mass of the beam per unit length, a the half-span of the beam, and $p_f(x, t)$ the back pressure of the fluid resisting the beam deformation.

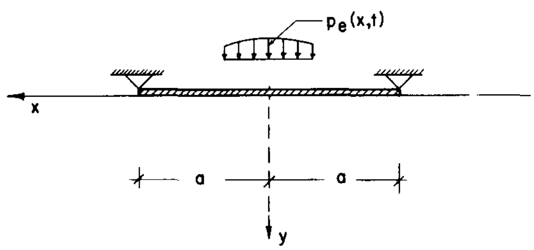

Fig. 1. Simply supported plate infinite in z-direction resting on a semi-infinite pool of fluid ($y \geq 0$).

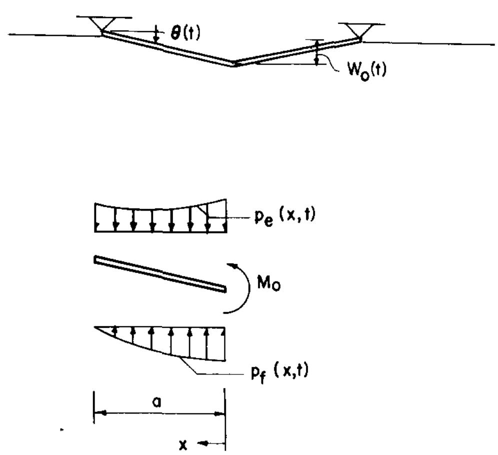

Fig. 2. Deformation mode of rigid ideally-plastic plate (beam). Loads acting on the beam.

The back pressure of the incompressible and irrotational fluid p_f is, according to the linearized (neglecting squares of the velocities) theory [8, 9] related to the fluid velocity potential $\phi(x, y, t)$

$$p_f(x, t) = -\gamma_f \frac{\partial \phi}{\partial t}\bigg|_{y=0} \tag{2}$$

where γ is the fluid density (constant) and $\mathbf{V} = \text{grad}\ \phi$ fluid velocity. In case of inviscid incompressible flow the velocity potential satisfies the two-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{3}$$

Neglecting the surface wave motion, one of the boundary conditions is

$$\phi = 0 \quad \text{at} \quad |x| \geq a, \quad y = 0. \tag{4}$$

Assuming, further, no separation at the fluid-beam interface†, the velocities of the fluid and beam particles in contact are identical

$$\frac{\partial \phi}{\partial y} = -\frac{\partial w}{\partial t} \quad \text{at} \quad |x| \leq a, \quad y = 0. \tag{5}$$

In addition, both the velocities and the velocity potential should be attenuating with the increase of distance from the origin, i.e.

$$\phi \to 0, \quad \frac{\partial \phi}{\partial n} \to 0 \quad \text{as} \quad n \to 0 \quad (n = x, y). \tag{6}$$

Finally, we assume the fluid to be initially at rest

$$\phi = \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad t = 0. \tag{7}$$

3. DETERMINATION OF THE FLUID BACK PRESSURE

Since the beam deformation mode $G(x)$ (Fig. 2) is stationary, i.e.

$$w(x, t) = W_0(t)\left(1 - \frac{x}{a} \operatorname{sign} x\right) = W_0(t) G(x). \tag{8}$$

It follows from eqn (3) and boundary conditions (4, 5) that the solution for $\phi(x, y, t)$ is separable

$$\phi(x, y, t) = \dot{W}_0(t) f(x, y) \tag{9}$$

where $\dot{W}_0(t)$ is obviously the velocity of the beam midspan cross section. Thus from eqn (2) it further follows that the back pressure is

$$p_f(x, t) = \gamma_f f(x, 0) \ddot{W}_0(t) = m_f(x) \ddot{W}_0(t) \tag{10}$$

where $m_f(x)$ is the so called apparent (added or virtual) mass associated with the resistance of the fluid displaced by the deforming structure.

In order to compute the apparent mass $m_f(x)$ it is necessary to solve for the distribution of velocity potential $f(x, y)$. From eqns (3), (4) and (5) it follows that $f(x, y)$ is the solution of the boundary value problem

†Note that this assumption is in the case much closer to reality than in the case of elastic vibration of structures resting on fluid [9].

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0 \tag{11}$$

$$f=0 \quad \text{for} \quad |x|\geq a, \quad y=0 \tag{12}$$

$$\frac{\partial f}{\partial y}=-G(x)=-\left(1-\frac{x}{a}\,\text{sign}\right)x \quad |x|\leq a, \quad y=0 \tag{13}$$

and

$$f\to 0, \quad \frac{\partial f}{\partial y}\to 0 \quad \text{as} \quad y\to\infty \tag{14}$$

$$f\to 0, \quad \frac{\partial f}{\partial x}\to 0 \quad \text{as} \quad x\to\infty. \tag{15}$$

Multiply eqn (11) by $\cos(\xi x)\mathrm{d}x$, integrate from 0 to ∞, and introduce the Fourier cosine transform[10]

$$\bar{f}_c(\xi, y)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty f(x,y)\cos(\xi x)\mathrm{d}x. \tag{16}$$

Note also that after integration by parts

$$\int \frac{\partial^2 f}{\partial x^2}\cos(\xi x)\mathrm{d}x=\left[\frac{\partial f}{\partial x}\cos(\xi x)+\xi f\sin(\xi x)\right]_0^\infty-\xi^2\int_0^\infty f\cos(\xi x)\mathrm{d}x=-\xi^2\int_0^\infty f\cos(\xi x)\mathrm{d}x \tag{17}$$

due to eqn (15) and since as a result of symmetry $u=\partial\phi/\partial x=0$ at $x=0$. Thus, the resulting equation reads

$$\frac{d^2\bar{f}_c}{dy^2}-\xi^2\bar{f}_c=0 \tag{18}$$

from where

$$\bar{f}_c(\xi, y)=A(a\xi)\,\mathrm{e}^{-\xi y}. \tag{19}$$

Due to eqn (14) only the part of solution attenuating with the increase in y exists. Using now the Fourier cosine inversion formula

$$f(x,y)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \bar{f}_c(\xi,y)\cos(\xi x)\,\mathrm{d}\xi \tag{20}$$

boundary conditions (12, 13) and the solution (19) it follows (for $y=0$)

$$\begin{aligned}&\int_0^\infty A(a\xi)\cos(\xi x)\mathrm{d}\xi=0 && |x|\geq a\\ &\int_0^\infty \xi A(a\xi)\cos(\xi x)\mathrm{d}\xi=\sqrt{\left(\frac{\pi}{2}\right)}\,G(x) && |x|\leq a.\end{aligned} \tag{21}$$

Introducing further new variables $u=a\xi$, $r=x/a$ and function

$$F(ar)=a^2G(ar)\sqrt{\frac{\pi}{2}} \tag{22}$$

eqns (21) can be rewritten in a standard dual integral equation form[11]

$$\begin{aligned} &\int_0^\infty A(u)\cos(ur)\,\mathrm{d}u = 0 \qquad &|r| \geq 1 \\ &\int_0^\infty uA(u)\cos(ur)\,\mathrm{d}u = F(ar) \qquad &|r| \leq 1. \end{aligned} \tag{23}$$

Solving eqn (23) for $A(u)$, It is possible to compute $\bar{f}_c(\xi, y)$ from eqns (19), $f(x, y)$ from eqn (20) and subsequently to determine the apparent mass m_f and back pressure p_f from eqn (10).

Solution of the governing system of dual integral equations

Once the governing system of equations is case into a form for which the solution exists the procedure becomes more or less routine. Writing,

$$\begin{aligned} \psi(r) &= \int_0^\infty uA(u)\cos(ru)\,\mathrm{d}u \\ \chi(r) &= \int_0^\infty A(u)\cos(ru)\,\mathrm{d}u \end{aligned} \tag{24}$$

and introducing representation

$$A(u) = \int_0^1 g(t)\,J_0(tu)\,\mathrm{d}t \tag{25}$$

(where J_0 is the Bessel function of zeroth order) the solution is derived[11] in the following form

$$\chi(r) = \int_r^1 \frac{g(t)\,\mathrm{d}t}{\sqrt{(t^2 - r^2)}} \quad \text{for} \quad |r| < 1 \tag{26}$$

where

$$g(t) = \frac{2t}{\pi}\int_0^t \frac{F(s)\,\mathrm{d}s}{\sqrt{(t^2 - s^2)}}. \tag{27}$$

From eqns (20) and (24b) it obviously follows that

$$f(x, 0) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{1}{a}\chi(x) \quad \text{for} \quad |x| \leq a. \tag{28}$$

Note that $\chi(r) = 0$ for $|r| > 1$ [11]. Solutions for $\psi(r)$ are also available but are of no interest for present purposes.

For the observed deformation mode eqn (13) it follows from eqns (22) and (27) that

$$g(t) = \frac{2t}{\pi}a^2\left(\frac{\pi}{2}\right)^{1/2}\left\{\int_0^t \frac{\mathrm{d}s}{(t^2 - s^2)^{1/2}} - \int_0^t \frac{s\,\mathrm{d}s}{(t^2 - s^2)^{1/2}}\right\} \tag{29}$$

$$\text{i.e.} \qquad g(t) = a^2\left(\frac{\pi}{2}\right)^{1/2} t\left(1 - \frac{2t}{\pi}\right). \tag{30}$$

Next, from eqns (26), (28) and (33)

$$f(r, 0) = a\frac{\pi - 1}{\pi}\sqrt{(1 - r^2)} - \frac{ar^2}{\pi}\log\frac{1 + \sqrt{(1 - r^2)}}{r} + C \tag{31}$$

where the integration constant $C = 0$ from the condition of vanishing velocity potential at the edge of the beam $f(1, 0) = 0$. The distribution of the apparent mass (eqns (10) and (31)) along one half of the beam $(0 \leq x \leq a)$ is plotted in Fig. 3.

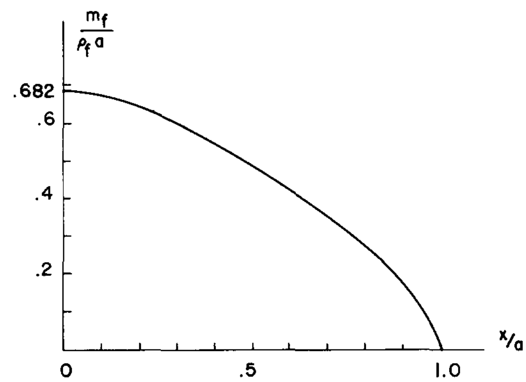

Fig. 3. Distribution of the apparent mass along the right-hand side half of the beam length.

4. SOLUTION OF THE BEAM PROBLEM

In order to solve the beam problem (1) it is necessary to compute the angular momentum M_f of the fluid back pressure about $x = a$,

$$M_f(t) = \int_0^a p_f(a-x)\,dx = \gamma_f \ddot{W}_0(t) a^2 \int_0^1 f(r,0)(1-r)\,dr. \tag{32}$$

Due to the presence of the second term in eqn (31), determination of $M_f(t)$ obviously involves some arduous integration. Instead, integrating eqn (26) with respect to r from 0 to 1

$$\int_0^1 \chi(r)\,dr = \int_0^1 dr \int_r^1 \frac{g(t)\,dt}{\sqrt{(t^2-r^2)}} \tag{33}$$

and changing the sequence of integration it follows that

$$\int_0^1 \chi(r)\,dr = \frac{\pi}{2}\int_0^1 g(t)\,dt. \tag{34}$$

Integration of eqn (30) is much simpler leading to

$$\int_0^1 \chi(r)\,dr = a^2\sqrt{\left(\frac{\pi}{2}\right)}\left(\frac{\pi}{4}-\frac{1}{3}\right). \tag{35}$$

In an identical way

$$\int_0^1 r\chi(r)\,dr = \int_0^1 tg(t)\,dt = a^2\sqrt{\left(\frac{\pi}{2}\right)}\left(\frac{1}{3}-\frac{1}{2\pi}\right). \tag{36}$$

Thus, from eqns (32), (35) and (36) it follows that the angular momentum of the fluid back pressure about $x = a$ is

$$M_f(t) = a^3\gamma_f \ddot{W}_0\left(\frac{\pi}{4}+\frac{1}{2\pi}-\frac{2}{3}\right) = 0{\cdot}2779 a^3 \gamma_f \ddot{W}_0. \tag{37}$$

Once the angular momentum of the back pressure is established, the problem reduces to a more-or-less routine problem of the dynamics of a rigid ideally-plastic beam extensively examined in the literature[1, 7].

With $\ddot{W}_0 = a\ddot{\theta}$, substitution of eqn (37) into eqn (1) leads finally to

$$-\frac{ma^3}{3}\ddot{\theta} = M_0 - \int_0^a p_e(a-x)\,dx \tag{38}$$

where the effective beam mass m consists of the actual beam mass m_s plus the apparent fluid mass m_f

$$m = km_s = (1 + 0{\cdot}8337\alpha)m_s \tag{39}$$

where

$$\alpha = \frac{a\gamma_f}{h\gamma_s} \tag{40}$$

with h being the beam height and $\gamma_s = m_s/h$ the beam density. Nondimensional coefficient k reflects the resistance of the fluid. Note that k can be of significant magnitude even for small density ratios γ_f/γ_s. Consider a typical steel beam resting on water, such that $a/h = 10$ and $\gamma_f/\gamma_s \simeq 0{\cdot}13$. Consequently, from eqns (39) and (40) $\alpha = 1{\cdot}3$ and $k = 2{\cdot}3$. For an aluminum beam resting on water $\gamma_f/\gamma_s = 0{\cdot}37$ and for the same $a/h = 10$ ratio, $k = 4{\cdot}7$. In other words, the apparent mass is more than doubled in case of steel and quadrupled in case of aluminum beams. When structural elements are enveloped by high density liquid metals (such as in new Liquid Metal Fast Breeder Reactors where liquid sodium is used as coolant) neglect of α in eqn (39) may result in an order of magnitude error.

Subsequent analysis is similar to analyses of rigid ideally-plastic beams in a vacuum [1]. As an illustration we consider the case of a simply supported beam subjected to uniformly distributed external pressure $p_e(t)$ over the entire span $-a \le x \le a$.

5. SIMPLY SUPPORTED BEAM SUBJECTED TO UNIFORMLY DISTRIBUTED LOAD

In the case of a uniformly distributed load p_e it follows from eqns (38) and (39)

$$-\frac{km_s a^3}{3}\ddot{\theta} = M_0 - \frac{pa^2}{2}. \tag{41}$$

Introducing parameter

$$\mu = \frac{p_e a^2}{2M_0} \tag{42}$$

selected such that $\mu = 1$ determines the fully-plastic moment for static loading (in a vacuum), eqn (41) can be recast into the following form

$$\frac{km_s a^3}{M_0}\ddot{\theta} = 3(\mu - 1). \tag{43}$$

Since $\ddot{\theta} \ge 0$ the lower limit for μ for which the deformation mode is triangular (Fig. 2) is $\mu = 1$ (as for a beam surrounded by a vacuum).

The admissible state of stress, corresponding to the triangular deformation mode (Fig. 2) is defined by

$$M(x,t)\begin{cases} < M_0 & \text{for} \quad 0 < x \le a \\ = M_0 & \text{for} \quad x = 0. \end{cases} \tag{44}$$

In order for a symmetric function to have only one maximum, its second derivative must not alternate in sign. Thus, the fluid backpressure p_f and the inertial force $m\ddot{w}$ combined should not exceed the magnitude of the external load p_e nowhere in the interval $0 \le |x| \le a$. Since both the fluid backpressure and the inertial force peak at midspan, the upper limit for μ for which the deformation mode is still triangular is obtained from the condition that the total load ($p_e + p_f$ + inertial force) equals zero at $x = 0$. In other words, since from eqns (10) and (31)

$$p_f(0,t) = \frac{\pi - 1}{\pi} a\gamma_f \ddot{W}_0(t). \tag{45}$$

It follows that at $x = 0$

$$p_e \geq \left(1 + \frac{\pi - 1}{\pi} \frac{a\gamma_f}{h\gamma_s}\right) m_s a\ddot{\theta} = (1 + k_1) m_s a\ddot{\theta}. \tag{46}$$

Using now eqns (43) and (46) it follows that the upper limit for μ is

$$\mu_{\max} = \frac{3}{3 - 2\frac{k}{k_1}}. \tag{47}$$

Since $k/k_1 \sim 1$ in most cases the permissible values of μ belong to the range $1 \leq \mu \leq 3$. For $\mu > 3$ the plastic hinge moves toward supports and the deformation mode is trapezoidal. For $\mu < 1$ the beam does not deform.

Solution of eqn (41) is routine[1] and we list it only for the sake of completeness. The final residual deflection W_{0f} is

$$\frac{km_s a^2}{M_0} W_{0f} = 3 \int_0^{t_f} I \, dt - \frac{3}{2} I_f^2 \tag{48}$$

where

$$I(t) = \int_0^t \mu \, dt. \tag{49}$$

The deformation terminates at $t = t_f$ defined by

$$t_f = I_f = \int_0^{t_f} \mu(t) dt. \tag{50}$$

6. SUMMARY AND CONCLUSIONS

This paper presents the closed form solution for the two-dimensional problem of a rigid ideally-plastic beam resting on a semi-infinite pool of irrotational and incompressible fluid. Although the derivation was conducted for a simply supported beam, the expression for the "total" masss m (eqn 39) is applicable to all beams developing the triangular deformation mode (eqn 8). Thus in addition to the simply supported beam, eqn (39) can be used for; (a) simply supported beam subjected to a force concentrated at midspan; (b) beam built-in at both ends and subjected to either uniformly distributed load or a force concentrated at midspan, etc.

In each case the limits on μ should be determined as shown in the previous section. Derived results also allow for a rational estimate of fluid resistance in case of more complex geometries.

Finally, it should be noted that while the modal techniques[12, 13] should be considered in analyses of more complicated problems, it will be necessary to investigate the relation (bound theorems) of the approximate to the exact solution in view of the fact that the load (backpressure p_f) depends itself on the assumed mode.

Derived formulae are, in summary, sufficiently simple to be readily used in engineering applications. Obtained results clearly demonstrate that the neglect of fluid presence will in most practical cases lead to totally erroneous conclusions.

It is understood that this derivation is based on a simplified analytical model for the fluid neglecting compressibility and the squares of velocity. Therefore, in the future further attempts should be made in order to assess the influence of neglected terms on the results as obtained in this paper.

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